

**RIGOROUS AND APPROXIMATE TREATMENT OF
LONG LINE TRANSMISSION PROBLEMS
BY HYPERBOLIC FUNCTIONS**

by
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I. INTRODUCTION.

It is generally agreed that in rigorous treatment of long transmission lines hyperbolic functions must be employed. In order to facilitate the application of these functions, charts and tables have been prepared; the last chart, due to Woodruff, covers the values of $\theta^2 = ZY$ between 0.05 and 0.36, corresponding to the lengths of aerial lines between approximately 100 and 300 miles at 60 c/s, because θ/L is practically constant and equal to 0.0018–0.0022 per mile at 60 c/s.

Almost all authors state that when using the expansions in series of hyperbolic functions it is possible to obtain any precision desired by taking a sufficient number of terms. They consider however the mathematical precision only; many of them (for example, Nesbitt, Wagner and Evans, etc.) give in their books examples of computation of successive terms in order to show that these terms diminish very rapidly, so that quite soon they do not affect the first, say four, significant figures.

A reference—rather vague—to the physical problem is found in G. D. McCann's excellent chapter No. 4 in the "Westinghouse Transmission and Distribution Reference Book" (1944). McCann fixes at 0.5 per cent. the mathematical precision of the developments, because "the resistance, inductance and capacitance of a line can rarely be known within 3 or 4 per cent. and probably never within 1 per cent. This is due to conductor sag, its variation with different spans and the variation that exists in conductor spacing, together with the effect of temperature upon conductor resistivity and sag."

In this paper we shall treat the physical aspect of the problem more comprehensively. We must consider two points:

(I) There is no physical significance in adding a new term of a development unless this term is large enough as compared to the maximum error committed in the value of the sum of preceding terms. We shall admit that it is so if a consecutive term is larger than 0.25 of the maximum error in the sum of the preceding terms. Of course, a value other than 0.25 may be chosen without affecting the generality of the treatment;

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(II) Even if it makes sense to add a new term in compliance with the consideration (I), its addition may not be worthwhile in view of the limited accuracy within which a given magnitude has to be known.

As a corollary of (II) we shall examine the problem under which conditions a line may be considered short, that is of negligible capacitance. The textbooks agree that an overhead line may be considered short when it is 20 or 30 miles long; some say that the effects of capacitance may be neglected whatever the length of line, if the voltage is lower than 40 kV. We shall see that the problem cannot be stated in this simple manner.

It is easy to see, applying to the term in θ^6 the same treatment that we shall use in respect to the terms in θ^2 and θ^4 , that the term in θ^6 may be omitted for present usual lengths of lines. Consequently we shall limit all developments to the terms in θ^4 and below.

II. ERRORS ON THE CONSTANTS OF A LINE.

Let, per mile,

r —the resistance, in ohms,

$l\omega$ —the inductive reactance, in ohms,

$c\omega$ —the capacitive susceptance, in mhos.

We shall neglect the conductance g , ordinarily very small. We have

$$\bar{Z} = Z/\gamma = \sqrt{r^2 + l^2\omega^2}L/\arctan l\omega/r,$$

$$\bar{Y} = Y/\pi/2 = c\omega L/90^\circ,$$

where L is the length of the line in miles.

The maximum errors on Z , γ and Y are

$$\frac{\Delta Z}{Z} = \frac{\Delta r}{r} \cos^2 \gamma + \frac{\Delta(l\omega)}{l\omega} \sin^2 \gamma + \frac{\Delta L}{L}, \quad (1)$$

$$\Delta \gamma = \left(\frac{\Delta(l\omega)}{l\omega} + \frac{\Delta r}{r} \right) \sin \gamma \cos \gamma, \quad (2)$$

$$\frac{\Delta Y}{Y} = \frac{\Delta(c\omega)}{c\omega} + \frac{\Delta L}{L}, \quad (3)$$

and are due to the variable atmospheric conditions, the variation of frequency and of the load, and the approximation of the formulas used for the calculation of the constants, or the errors in the measurements of these constants.

III. THE EQUIVALENT π CIRCUIT.

We have

$$\bar{Z}_e = \bar{Z} \frac{\sinh \bar{\theta}}{\bar{\theta}} = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} + \frac{\bar{\theta}^4}{120} + \frac{\bar{\theta}^6}{5040} + \dots \right), \quad (4)$$

$$\bar{Y}_e = \frac{\bar{Y} \tanh (\bar{\theta}/2)}{(\bar{\theta}/2)} = \frac{\bar{Y}}{2} \left(1 - \frac{\bar{\theta}^2}{12} + \frac{\bar{\theta}^4}{120} - \frac{17\bar{\theta}^6}{20140} + \dots \right). \quad (5)$$

A. The Development of $\bar{Z}_e$.

We have, limiting the development to the term in $\bar{\theta}^4$,

$$\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} + \frac{\bar{\theta}^4}{120} \right) \quad (6)$$

and the projections of $\bar{Z}_e$, M on $\bar{Z}$ and N on the normal to $\bar{Z}$, are (Fig. 1):

$$M = Z \left(1 - \frac{\theta^2}{6} \sin \gamma - \frac{\theta^4}{120} \cos 2\gamma \right), \quad (7)$$

$$N = Z \left(\frac{\theta^2}{6} \cos \gamma - \frac{\theta^4}{120} \sin 2\gamma \right). \quad (8)$$

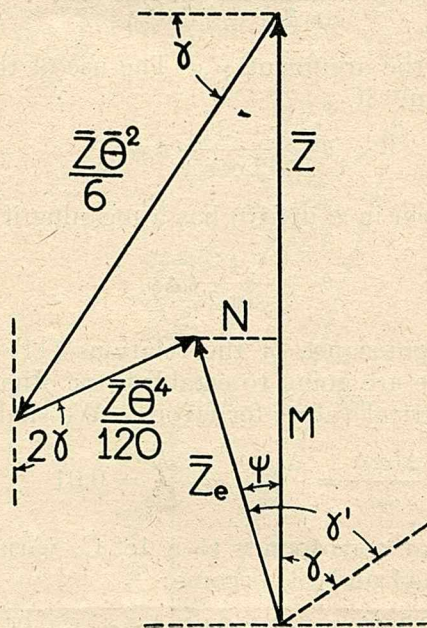


FIG. 1.

Hence

$$\tan \psi = N/M \cong \frac{\theta^2}{6} \cos \gamma + \frac{\theta^4}{180} \sin 2\gamma \cong \psi.$$

The modulus and argument of $\bar{Z}_e$ are

$$\begin{aligned} Z_e &= \frac{M}{\cos \psi} \cong M(1 + \frac{1}{2} \tan^2 \psi) \\ &\cong Z \left[1 - \frac{\theta^2}{6} \sin \gamma - \frac{\theta^4}{120} (\cos 2\gamma - \frac{5}{3} \cos^2 \gamma) \right], \end{aligned} \quad (9)$$

$$\gamma' = \gamma + \psi \cong \gamma + \frac{\theta^2}{6} \cos \gamma + \frac{\theta^4}{180} \sin 2\gamma. \quad (10)$$

If the use of the second term, $(Z\theta^2 \sin \gamma)/6$, in (9) has to have any physical meaning, we must have, according to the consideration (I):

$$\frac{1}{4}\Delta Z \leq (Z\theta^2 \sin \gamma)/6; \text{ hence } \theta^2 \geq \frac{3}{2 \sin \gamma} \frac{\Delta Z}{Z}. \quad (11)$$

In examining the use of the third term, we should compare it with the maximum error in $Z - (Z\theta^2 \sin \gamma)/6$, but in an analysis of errors it will be sufficiently correct to compare it with the maximum error in Z only, $(Z\theta^2 \sin \gamma)/6$ being relatively small with respect to Z . Consequently, the use of the term in θ^4 in (9) has no physical meaning unless

$$\theta^4 \geq \frac{30}{|\cos 2\gamma - \frac{5}{3} \cos^2 \gamma|} \frac{\Delta Z}{Z}. \quad (12)$$

Let us consider now the argument γ' . The use of the second term in (10) has a meaning only if

$$\theta^2 \geq \frac{3}{2 \cos \gamma} \Delta \gamma \quad (13)$$

and the use of the term in θ^4 in (10) has a meaning if

$$\theta^4 \geq \frac{45}{\sin 2\gamma} \Delta \gamma. \quad (14)$$

To see the actual significance of the relations (11)–(14), and of the similar ones which we are going to establish for other magnitudes, we have to assume numerical values for errors. We shall admit

$$\frac{\Delta(c\omega)}{c\omega} = \frac{\Delta(l\omega)}{l\omega} = \frac{\Delta L}{L} = 0.01$$

and $\Delta r/r = 0.06$, what corresponds to a 15° C. variation in temperature. The relations (11 and 13) become:

$$\theta^2 \geq \frac{3 \cdot 0.06 \cos^2 \gamma + 0.01 \sin^2 \gamma + 0.01}{2 \sin \gamma}, \quad (11a)$$

$$\theta^2 \geq 0.105 \sin \gamma. \quad (13a)$$

The relations (11a) and (13a) show that the lengths L' (modulus) and L'' (argument) for which the terms in θ^2 have to be considered in (9) and (10), respectively, depend on γ . Making $\theta = 2 \cdot 10^{-3} L$, we find

modulus (11a): $L' = 85$ miles for $\gamma \cong 90^\circ$, and 155 miles for $\gamma = 45^\circ$;
argument (13a): $L'' = 160$ miles for $\gamma \cong 90^\circ$, and 135 miles for $\gamma = 45^\circ$.

Similarly, the relation (12) gives

$$L' = 450 \text{ miles for } \gamma \cong 90^\circ, \text{ and } 600 \text{ miles for } \gamma = 45^\circ.$$

The relation (14) gives $L'' = 550$ miles independently of γ .

Let us consider now the argument of $\bar{Y}_e$, $(90^\circ - \psi)$. In view that we assumed $g = 0$, the term in θ^2 in (16) must always be considered, as far as only the consideration (I) is concerned.

The second term in (16) has to be considered only if

$$\theta^2 \geq \frac{15}{7 \sin \gamma} \left(\frac{\Delta r}{r} + \frac{\Delta L}{L} + \frac{\Delta Y}{Y} \right) \quad (19)$$

c.

Until now we have considered $\bar{Z}_e$ and $\bar{Y}_e$ alone, but generally it is necessary to consider them in combination with themselves and with other magnitudes. The conclusions obtained before may not be valid, because of possibility of compensation and of the presence of new terms.

We are interested in obtaining correct values of the terms in θ^2 and θ^4 in the developments corresponding to various magnitudes. Note

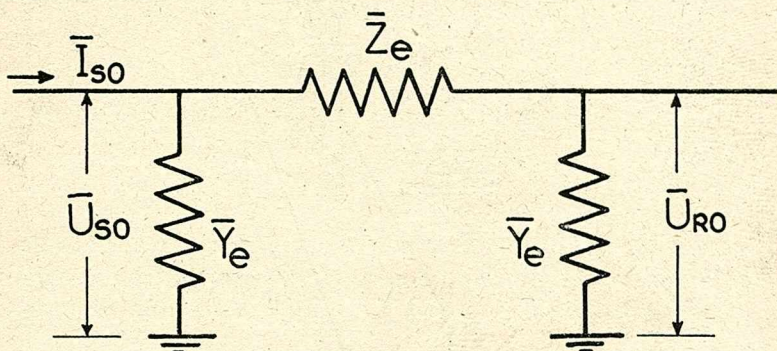


FIG. 3.

that for expression of a form $\bar{Z}_e^a \bar{Y}_e^b$, where a and b are positive integers or zero, it will be sufficient to make $\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} \right)$ and $\bar{Y}_e = \frac{Y}{2} \left(1 - \frac{\bar{\theta}^2}{12} \right)$; the coefficient of the term in $\bar{\theta}^2$ will be always exact, and the one of the term in θ^4 will be exact if $a > 0$ and $b > 0$; if $a = 0$ or $b = 0$, this coefficient will be too small in $(a + b)/120$.

If we make $\bar{Z}_e = \bar{Z}$ and $\bar{Y}_e = \bar{Y}$, the coefficient of $\bar{\theta}^2$ will be exact only if $a > 0$ and $b > 0$.

IV. LINE AT NO LOAD. RELATION OF THE VOLTAGES.

We have (Fig. 3):

$$\frac{\bar{U}_{s0}}{\bar{U}_{R0}} = 1 + \bar{Z}_e \bar{Y}_e = \bar{A} = \bar{A}/\delta, \quad (20)$$

where $\bar{A}$ is one of the constants $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D} = \bar{A}$ of the equivalent four terminal network.

Let us make in (20) $\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} \right)$ and $\bar{Y}_e = \frac{\bar{Y}}{2} \left(1 - \frac{\bar{\theta}^2}{12} \right)$, we obtain

$$\bar{A} = 1 + \frac{\bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{24}, \quad (21)$$

where the coefficient of $\bar{\theta}^4$ is exact, according to what we said above. The modulus is

$$A = \frac{U_{SO}}{U_{RO}} = 1 - \frac{\theta^2}{2} \sin \gamma - \frac{\theta^4}{24} (\cos 2\gamma - 3 \cos^2 \gamma) \quad (22)$$

and there is a meaning in using the term in θ^4 if this term is larger than 0.25 of the error in the term in θ^2 , that is, if

$$\theta^2 \geq \frac{3 \sin \gamma}{1 + \cos^2 \gamma} \left[\frac{\Delta(l\omega)}{l\omega} + \frac{\Delta L}{L} + \frac{\Delta Y}{Y} \right]. \quad (23)$$

If $\frac{\Delta(l\omega)}{l\omega} + \frac{\Delta L}{L} + \frac{\Delta Y}{Y} = 0.04$, for $\gamma \cong 90^\circ$ the length must be greater than 175 miles, and for $\gamma = 45^\circ$ greater than 120 miles.

The value of A is obtained with great accuracy by means of (22); for example, if $L = 120$ miles, we may neglect the term in θ^4 (which means here to make $\bar{Z}_e = \bar{Z}$, $\bar{Y}_e = \bar{Y}/2$, and $\cos \delta = 1$) and the maximum error on A will be between ϵ and 1.25ϵ , with

$$\epsilon = \frac{\theta^2 \sin \gamma}{2} \left[\frac{\Delta(l\omega)}{l\omega} + \frac{\Delta L}{L} + \frac{\Delta Y}{Y} \right]. \quad (24)$$

For $L = 120$ miles, and $\frac{\Delta(l\omega)}{l\omega} + \frac{\Delta L}{L} + \frac{\Delta Y}{Y} = 0.04$, $\epsilon \cong 0.001$. But normally it is sufficient that A have a much less accuracy. Indeed, if we determine A by measuring U_{SO} and U_{RO} on the respective switch-board voltmeters, it is impossible to expect to obtain A within a maximum error less than 2 per cent. In such a case it is useless to take into account the term $(\theta^2 \sin \gamma)/2$ if it is smaller than 0.02, that is, if $\sin \gamma \cong 1$, for lines of less than 100 miles. Then, under the conditions just exposed a line up to 100 miles has to be considered as "short."

V. LINE AT NO LOAD. CURRENT AND POWER AT THE SENDING END.

A. Current at the Sending End.

We have (Fig. 3)

$$\frac{\bar{I}_{SO}}{\bar{U}_{SO}} = \bar{Y}_e \frac{2 + \bar{Z}_e \bar{Y}_e}{1 + \bar{Z}_e \bar{Y}_e} = \frac{\bar{C}}{\bar{A}}. \quad (25)$$

Let us make $\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} \right)$ and $\bar{Y}_e = \frac{\bar{Y}}{2} \left(1 - \frac{\bar{\theta}^2}{12} \right)$; we find

$$\frac{\bar{I}_{so}}{\bar{U}_{so}} = \bar{Y} \left(1 - \frac{\bar{\theta}^2}{3} + \frac{\bar{\theta}^4}{8} \right) \quad (26)$$

with an error in the coefficient of $\bar{\theta}^4$ equal to $1/120$; the exact value of this coefficient is $2/15$.

The modulus of $\bar{I}_{so}/\bar{U}_{so}$ is

$$\frac{I_{so}}{U_{so}} = Y \left[1 + \frac{\theta^2}{3} \sin \gamma - \frac{2}{15} \theta^4 (\cos 2\gamma - \frac{5}{12} \cos^2 \gamma) \right] \quad (27)$$

and there only is a meaning in using the term in θ^4 if

$$\theta^4 \geq \frac{15}{8} \frac{1}{|\cos 2\gamma - \frac{5}{12} \cos^2 \gamma|} \frac{\Delta Y}{Y}, \quad (28)$$

that is, if $\Delta Y/Y = 0.02$, 220 miles for $\gamma \cong 90^\circ$ and 320 miles for $\gamma = 45^\circ$. For any length shorter than defined by equation (28) we may write

$$\frac{I_{so}}{U_{so}} = Y \left(1 + \frac{\theta^2}{3} \sin \gamma \right) \quad (29)$$

with a maximum error between $\Delta Y/Y$ and $1.25 \Delta Y/Y$.

To neglect the term in θ^4 is not equivalent here to make $\bar{Z}_e = \bar{Z}$ and $\bar{Y}_e = \bar{Y}/2$. If we limit the developments of $\bar{Z}_e$ and $\bar{Y}_e$ to their first terms, we find

$$\frac{I_{so}}{U_{so}} = Y \left(1 + \frac{\theta^2}{4} \sin \gamma \right) \quad (30)$$

with a supplementary negative error $(\theta^2 \sin \gamma)/12$.

If it is sufficient to obtain I_{so}/U_{so} with an even less accuracy, we may write, neglecting $\bar{Z}_e \bar{Y}_e$ compared to 1,

$$\frac{I_{so}}{U_{so}} = 2Y_e = Y \left(1 + \frac{\theta^2}{12} \sin \gamma \right) \quad (31)$$

with a supplementary negative error $(\theta^2 \sin \gamma)/4$, or

$$\frac{I_{so}}{U_{so}} = Y \quad (32)$$

with a supplementary negative error of $(\theta^2 \sin \gamma)/3$. We have now to investigate when a line may be considered as "short" from the point of view of the no-load current. In order to give a physical significance to this problem we must refer I_{so} to the load current I_n . Assume that the line is transmitting its "natural" power, magnitude introduced by Rüdenberg to characterize a load for which the line operates at nearly

its maximum efficiency. Under these conditions

$$P = 3U_R^2 / \sqrt{\frac{Z \sin \gamma}{Y}} \quad (33)$$

and hence

$$I_n = U_R \cos \varphi_R / \sqrt{\frac{Z \sin \gamma}{Y}}, \quad (34)$$

where $\cos \varphi_R$ is the power factor of the receiver.

From (32) and (34)

$$\frac{I_{so}}{I_n} = \frac{U_{so}}{U_R} \frac{\sqrt{\sin \gamma}}{\cos \varphi_R} \theta \cong \frac{U_{so}}{U_R} \frac{\sqrt{\sin \gamma}}{\cos \varphi_R} 2.10^{-3} L \quad (35)$$

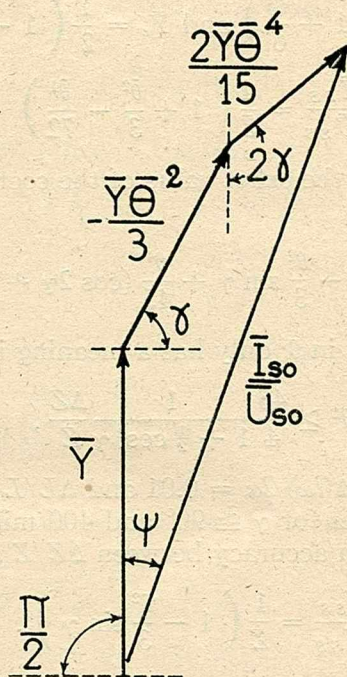


FIG. 4.

with L in miles. $U_{so}\sqrt{\sin \gamma}/U_R \cos \varphi_R$ is nearly equal to 1; consequently we find that to call a line of a length less than 20 miles "short" means to neglect a no-load current equal approximately to 5 per cent of the full-load current.

B. Power at the sending end.

We have, calling ψ the angle between $\bar{I}_{so}/\bar{U}_{so}$ and $\bar{Y}$ (Fig. 4):

$$\frac{P_{so}}{U_{so}^2} = \frac{I_{so}}{U_{so}} \sin \psi = Y \left(\frac{\theta^2}{3} \cos \gamma + \frac{2\theta^4}{15} \sin 2\gamma \right) \quad (36)$$

and the use of the term in θ^4 has a meaning if

$$\theta^2 \geq \frac{5}{16 \sin \gamma} \left(\frac{\Delta r}{r} + \frac{\Delta L}{L} + \frac{2\Delta Y}{Y} \right), \quad (37)$$

that is, if $\Delta r/r + \Delta L/L + 2\Delta Y/Y = 0.11$ and $\sin \gamma \cong 1$, for lengths greater than 90 miles.

VI. LINE IN SHORT-CIRCUIT. CURRENT AT THE SENDING END.

We have

$$\frac{\bar{I}_{ss}}{\bar{U}_{ss}} = \frac{1 + \bar{Z}_e \bar{Y}_e}{\bar{Z}_e} = \frac{\bar{A}}{\bar{B}}. \quad (38)$$

Let us make $\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} \right)$ and $\bar{Y}_e = \frac{\bar{Y}}{2} \left(1 - \frac{\bar{\theta}^2}{12} \right)$; we obtain

$$\frac{\bar{I}_{ss}}{\bar{U}_{ss}} = \frac{1}{\bar{Z}} \left(1 + \frac{\bar{\theta}^2}{3} - \frac{\bar{\theta}^4}{72} \right) \quad (39)$$

with an error of $\bar{\theta}^4/120$; the exact value of the coefficient of $\bar{\theta}^4$ is $-1/45$. The modulus is

$$\frac{I_{ss}}{U_{ss}} = \frac{1}{Z} \left[1 - \frac{\theta^2}{3} \sin \gamma + \frac{\theta^4}{45} (\cos 2\gamma + \frac{5}{2} \cos^2 \gamma) \right] \quad (40)$$

and the use of the term in θ^4 only has a meaning if

$$\theta^4 \geq \frac{45}{4} \frac{1}{1 + \frac{1}{2} \cos^2 \gamma} \frac{\Delta Z}{Z}, \quad (41)$$

that is, if $\Delta r/r = 0.06$, $\Delta(l\omega)/l\omega = 0.01$ and $\Delta L/L = 0.01$, if the length is greater than 350 miles for $\gamma \cong 90^\circ$ and 400 miles for $\gamma \cong 45^\circ$. We may then write, with an accuracy between $\Delta Z/Z$ and $1.25 \Delta Z/Z$,

$$\frac{I_{ss}}{U_{ss}} = \frac{1}{Z} \left(1 - \frac{\theta^2}{3} \sin \gamma \right). \quad (42)$$

If we may tolerate a larger error, we can make $\bar{Z}_e = \bar{Z}$ and $\bar{Y}_e = \bar{Y}/2$, obtaining

$$\frac{I_{ss}}{U_{ss}} = \frac{1}{Z} \left(1 - \frac{\theta^2}{2} \sin \gamma \right), \quad (43)$$

the supplementary error being negative and equal to $(\theta^2 \sin \gamma)/6$.

We may also simplify (38) neglecting $\bar{Z}_e \bar{Y}_e$ compared to 1; we find

$$\frac{I_{ss}}{U_{ss}} = \frac{1}{Z_e} = \frac{1}{Z} \left(1 + \frac{\theta^2}{6} \sin \gamma \right) \quad (44)$$

with a supplementary positive error $(\theta^2 \sin \gamma)/2$.

Finally, let us observe that the simplest expression for I_{ss}/U_{ss} ,

$$\frac{I_{ss}}{U_{ss}} = \frac{1}{Z}, \quad (45)$$

is equivalent to the consideration of the line as "short." It leads to a supplementary positive error of $(\theta^2 \sin \gamma)/3$. If we want to find I_{ss}/U_{ss} within a total accuracy of 10 per cent., with $\Delta r/r = 0.06$, $\Delta(l\omega)/l\omega = 0.01$, $\Delta L/L = 0.01$ and $\sin \gamma \cong 1$, the length should be less than about 240 miles. For 300 miles, using (45), the total error would be = 14 per cent. if $\gamma \cong 90^\circ$, 13 per cent. of $\gamma = 45^\circ$.

VII. LINE AT LOAD.

A. Relation of the Voltages.

We have

$$\frac{\bar{U}_s}{\bar{U}_R} = 1 + \bar{Z}_e \bar{Y}_e + \bar{Z}_e \bar{Y}_c = \bar{A} + \bar{B} \bar{Y}_c, \quad (46)$$

where $\bar{Y}_c$ is the admittance of the load.

Let us make $\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} \right)$ and $\bar{Y}_e = \frac{\bar{Y}}{2} \left(1 - \frac{\bar{\theta}^2}{12} \right)$; we find

$$\frac{\bar{U}_s}{\bar{U}_R} = 1 + \bar{\theta}^2 \left(\frac{1}{2} + \frac{\bar{Y}_c}{\bar{Y}} \right) + \frac{\bar{\theta}^4}{6} \left(\frac{1}{4} + \frac{\bar{Y}_c}{\bar{Y}} \right). \quad (47)$$

If the power transmitted is the natural power of the line, we have

$$Y_c = \sqrt{\frac{Y}{Z \sin \gamma}} \cos \varphi_R, \quad \text{and} \quad \frac{Y_c}{Y} = \frac{\cos \varphi_R}{\sqrt{\sin \gamma} \theta}. \quad (48)$$

For $\cos \varphi_R \cong 1 \cong \sin \gamma$, and a length of 300 miles, $Y_c/Y = 1.6$, so that we may admit, only for the purpose of discussion of errors,

$$\frac{1}{2} + \frac{\bar{Y}_c}{\bar{Y}} \cong \frac{1}{4} + \frac{\bar{Y}_c}{\bar{Y}} \cong \frac{\bar{Y}_c}{\bar{Y}} = \frac{Y_c}{Y} / -\varphi_R - \pi/2.$$

For the same purpose we may neglect the correction of the cosine of the angle between 1 and $\bar{U}_s/\bar{U}_R$, and find the following approximate expression for the modulus of $\bar{U}_s/\bar{U}_R$:

$$\frac{U_s}{U_R} = 1 + Z Y_c \cos(\gamma - \varphi_R) - \frac{\theta^2 Z Y_c}{6} \sin(2\gamma - \varphi_R), \quad (49)$$

and the use of the third term in (49) only has a meaning if

$$\theta^2 \geq \frac{3}{2 \sin(2\gamma - \varphi_R)} \left[\frac{\Delta L}{L} \cos(\gamma - \varphi_R) + \frac{\Delta r}{r} \cos \gamma \cos \varphi_R + \frac{\Delta(l\omega)}{l\omega} \sin \gamma \sin \varphi_R \right], \quad (50)$$

for example, if $\gamma \cong 70^\circ$, $\varphi_R \cong 25^\circ$, $\Delta r/r = 0.06$, $\Delta(l\omega)/l\omega = 0.01 = \Delta L/L$, for $L \geq 110$ miles. For lengths less than the one defined by (50), we may make $\bar{Z}_e = \bar{Z}$ and $\bar{Y}_e = \bar{Y}/2$ in (46), which is equivalent to neglecting the third term in (47). For greater lengths all depends on the accuracy within which we want to know the relation U_S/U_R . For a line of 300 miles, to neglect the term in θ^4 in (49) is equivalent to a positive error equal to $0.02 Y_c/Y$, that is, about 3 per cent. if the natural power is transmitted. It must be observed that this error is proportional to the fourth power of the length, L^4 .

B. Relation of the Currents.

We have

$$\frac{\bar{I}_S}{\bar{I}_R} = 1 + \bar{Z}_e \bar{Y}_e + \bar{Z}_c \bar{Y}_e (2 + \bar{Z}_e \bar{Y}_e) = \bar{A} + \bar{C} \bar{Z}_c, \quad (51)$$

where $\bar{Z}_c$ is the impedance of the load. Let us make $\bar{Z}_e = \bar{Z} \left(1 + \frac{\bar{\theta}^2}{6} \right)$ and $\bar{Y}_e = \frac{\bar{Y}}{2} \left(1 - \frac{\bar{\theta}^2}{12} \right)$; we find

$$\frac{\bar{I}_S}{\bar{I}_R} = 1 + \bar{\theta}^2 \left(\frac{1}{2} + \frac{\bar{Z}_c}{\bar{Z}} \right) + \frac{\bar{\theta}^4}{6} \left(\frac{1}{4} + \frac{\bar{Z}_c}{\bar{Z}} \right). \quad (52)$$

If the line transmits the natural power

$$\frac{Z_c}{Z} = \frac{\sqrt{\sin \gamma}}{\cos \varphi_R} \frac{1}{\theta}, \quad (53)$$

that is, for a line of 300 miles, $\sin \gamma \cong 1$ and $\cos \varphi_R \cong 1$, $\bar{Z}_c/\bar{Z} = 1, 6/\varphi_R - \gamma$, so that we may still make for the purpose of the discussion of errors

$$\frac{1}{2} + \frac{\bar{Z}_c}{\bar{Z}} \cong \frac{1}{4} + \frac{\bar{Z}_c}{\bar{Z}} \cong \frac{\bar{Z}_c}{\bar{Z}}.$$

Furthermore we may neglect for the same purpose the correction of the cosine of the angle between 1 and $\bar{I}_S/\bar{I}_R$, and find the following approximate expression for the modulus of $\bar{I}_S/\bar{I}_R$:

$$\frac{I_S}{I_R} = 1 - Z_c Y \sin \varphi_R - \frac{Z_c Y \theta^2}{6} \cos (\gamma + \varphi_R). \quad (54)$$

Here the third term has a meaning if

$$\theta^2 \geq \frac{3 \Delta Y}{2 Y} \frac{\sin \varphi_R}{|\cos (\gamma + \varphi_R)|}, \quad (55)$$

that is, if $\Delta Y/Y = 0.02$, $\gamma \cong 70^\circ$ and $\varphi_R \cong 25^\circ$, for lengths greater than 200 miles. Furthermore it may often be neglected because we only need to know I_S/I_R within a limited accuracy. For a line of 300 miles,

and for $\cos(\gamma + \varphi_R)$ as high as 0.5 the error in neglecting the third term, thus making $\bar{Z}_e = \bar{Z}$ and $\bar{Y}_e = \bar{Y}/2$ in (51), is positive and lower than 2 per cent.

VIII. LINE AT LOAD. POWER CIRCLES AT SENDING AND RECEIVING ENDS.

We have

$$\frac{\bar{P}_S}{U_S^2} = \frac{1 + \bar{Z}_e \bar{Y}_e}{\bar{Z}_e} - \frac{\bar{U}_R}{\bar{U}_S} \frac{1}{\bar{Z}_e} = \frac{\bar{A}}{\bar{B}} - \frac{1}{\bar{B}} \frac{\bar{U}_R}{\bar{U}_S} \quad (56)$$

and

$$\frac{\bar{P}_R}{U_R^2} = \frac{\bar{U}_S}{\bar{U}_R} \frac{1}{\bar{Z}_e} - \frac{1 + \bar{Z}_e \bar{Y}_e}{\bar{Z}_e} = \frac{\bar{U}_S}{\bar{U}_R} \frac{1}{\bar{B}} - \frac{\bar{A}}{\bar{B}}. \quad (57)$$

The modulus of the relation $\bar{A}/\bar{B}$ has been studied in VI. We found that in practically all cases it is sufficient to take

$$\frac{\bar{A}}{\bar{B}} = \frac{1}{\bar{Z}} \left(1 - \frac{\theta^2}{3} \sin \gamma \right) = \frac{1}{\bar{Z}} \left(1 - \frac{l\omega LY}{3} \right). \quad (58)$$

The argument of $\bar{A}/\bar{B}$ is

$$\gamma'' = -\gamma + \frac{\theta^2}{3} \cos \gamma (1 + \frac{7}{15} \theta^2 \sin \gamma), \quad (59)$$

where the use of the term in θ^4 only has a meaning if

$$\theta^4 \geq \frac{45}{28} \left[\frac{\Delta(l\omega)}{l\omega} + \frac{\Delta r}{r} \right], \quad (60)$$

that is, for $\Delta(l\omega)/l\omega + \Delta r/r = 0.07$, for $L > 280$ miles.

We may then write for all usual cases

$$\gamma'' = -\gamma + \frac{\theta^2}{3} \cos \gamma = -\gamma + \frac{rLY}{3}. \quad (61)$$

For the radii, $\frac{U_S}{U_R} \frac{1}{\bar{Z}_e}$ and $\frac{U_R}{U_S} \frac{1}{\bar{Z}_e}$, according to the study done in III, is practically always sufficient to write

$$\bar{Z}_e = \bar{Z} \left(1 - \frac{l\omega LY}{6} \right). \quad (62)$$

The zero position of the radii, for which $\bar{U}_S$ and $\bar{U}_R$ are in phase, is defined by

$$-\gamma' = -\gamma - \frac{\theta^2}{6} \cos \gamma = -\gamma - \frac{rLY}{6} \quad (63)$$

The relations (58), (61), (62) and (63) determine the centers and radii of the circles of power with all the accuracy which has physical meaning for lines up to 300 miles, and without any necessity of using the hyperbolic functions.

CONCLUSIONS

We have shown that because of the errors in the constants of the line, generally there is no physical meaning in using more than two terms in the developments of $\bar{Z}_e$ and $\bar{Y}_e$. Furthermore, because it is of no use to try to determine the magnitudes beyond a certain accuracy, in many cases it is perfectly correct to limit the expansions of $\bar{Z}_e$ and $\bar{Y}_e$ to the first terms, that is $\bar{Z}$ and $\bar{Y}/2$, respectively.

The definition of a line as a "short" (of negligible capacitance) if the length is less than a given value, is not rational. It is necessary to specify the criterion employed. We have seen that a line is "short" if

$$L \leq 20 \text{ miles} \quad \text{by the condition of neglecting } I_{SO} \approx 0.05 I_n, \quad (\text{V})$$

$$L \leq 100 \text{ miles} \quad \text{by the condition } U_{SO}/U_{RO} \geq 0.98, \quad (\text{IV})$$

$$L \leq 240 \text{ miles} \quad \text{by the condition } I_{SS}/U_{SS} \text{ exact} \\ \text{within 10 per cent.}, \quad (\text{VI})$$

for specified values of γ , $\Delta r/r$, $\Delta(l\omega)/l\omega$, $\Delta L/L$.

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